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TRANSFORMATION GROUPS APPLIED TO ORDINARY DIFFERENTIAL EQUATIONS.

By DR. JAMES M. PAGE, Cobham, Va.

We propose in the following article to show in as elementary a manner as possible how Transformation Groups may be utilized in integrating certain differential equations of the first order. We shall limit ourselves to two variables; but the most important of the results may be extended at once to n variables.

The matter is, for the most part, Lie's; and it was obtained chiefly from the writer's notes on Lie's lectures in Leipzig in 1887–8.

It was shown in a previous artice (ANNALS, Vol. VIII, No. 4) that, if

$$\omega(x, y) = \text{const.} \tag{1}$$

be any family of ∞^1 curves in the plane; and if there is given an infinitesimal transformation of the form

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y},$$

then the curves (1) are said to *admit of* the transformation Uf, when, and only when,

$$U(\omega) = \Omega(\omega). \tag{2}$$

The criterion (2) holds also for the group of one member, or the G_1, which is *generated* by Uf; thus it is said that the family of curves (1) *admits of* the G_1, Uf, when the equation (2) is satisfied. It was also shown that

$$U(\omega) = \xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial y};$$

and if this expression becomes identically zero, the curves (1) are called the *loci* of the G_1, Uf.

The family of ∞^1 curves (1) is, of course, also represented by a differential equation of the first order, of the form

$$X(x, y)\,dy - Y(x, y)\,dx = 0, \tag{3}$$

of which $\omega(x, y)$ is the integral.

Hence, ω must satisfy a linear partial differential equation of the first order, of the form

$$X\frac{\partial \omega}{\partial x} + Y\frac{\partial \omega}{\partial y} = 0;$$

and every function of ω, $\Phi(\omega)$, must also satisfy this equation, and be an integral of (3).

If we perform the transformation Uf on $\Phi(\omega)$, we find

$$U(\Phi) \equiv \frac{d\Phi}{d\omega} \cdot U(\omega) \equiv \frac{d\Phi}{d\omega} \cdot \Omega(\omega) .$$

If the curves (1) are not the *loci* of the G_1, Uf,—that is, if $\Omega(\omega)$ is not zero,—we can evidently always choose Φ as a function of u in such a manner that

$$\frac{d\Phi}{d\omega} \cdot \Omega(\omega) = 1 .$$

Let us suppose Φ so chosen ; then we have,

$$X\frac{\partial\Phi}{\partial x} + Y\frac{\partial\Phi}{\partial y} = 0 ,$$

$$U(\Phi) \equiv \xi\frac{\partial\Phi}{\partial x} + \eta\frac{\partial\Phi}{\partial y} = 1 .$$

Hence,

$$\frac{\partial\Phi}{\partial x} = -\frac{Y}{X\eta - Y\xi}, \quad \frac{\partial\Phi}{\partial y} = -\frac{X}{X\eta - Y\xi}.$$

Then

$$d\Phi \equiv \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy = \frac{Xd\eta - Ydx}{X\eta - Y\xi} .$$

The last expression must be a *complete differential ;* that is,

$$U \equiv \frac{1}{X\eta - Y\xi}$$

must be an integrating factor of (3).

A differential equation is said to *admit of* a transformation, when, after carrying out the transformation, the differential equation preserves in the new variables its original form, with the exception of a factor which may be canceled. Thus the equation

$$X(x, y)\,dy - Y(x, y)\,dx = 0 \tag{3}$$

is said to admit of the transformation

$$x_1 = \varphi(x, y) , \quad y_1 = \psi(x, y) ,$$

if, in the new variables, (3) assumes the form

$$\rho(x_1, y_1)\,\{X(x_1, y_1)\,dx_1 - Y(x_1, y_1)\,dy_1\} = 0 ,$$

ρ being any function of x_1, y_1.

It is quite easy to show by a rigid proof the almost obvious fact that a differential equation (3) will admit of the infinitesimal transformation Uf,—or, as we may say, of the G_1, Uf,—when and only when the family of the ∞^1 integral curves of (3) admits of Uf.

Hence we find the important result:

If a given differential equation of the first order in x and y,

$$X(x, y)\,dy - Y(x, y)\,dx = 0\,;$$

admits of the G_1, Uf, where the infinitesimal transformation of the G_1 has the general form

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y},$$

then

$$U = \frac{1}{X\eta - Y\xi}$$

is an integrating factor of the differential equation.

The condition must be fulfilled here that $X\eta - Y\xi$ is not identically 0. It can be readily seen that if $X\eta - Y\xi = 0$, then the curves (1) are the *loci* of the G_1, Uf; and we say in this case that the transformation Uf is *trivial* as regards the differential equation, since it tells us nothing new.

It is, of course, necessary to develop a practical criterion to tell *when* a given differential equation (3) will admit of a given infinitesimal transformation, Uf.

If the differential equation be taken in the form

$$Xdy - Ydx = 0\,, \tag{3}$$

then to find an integral of (3) is the same problem as to find a solution of the linear partial differential equation,

$$Af = X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = 0\,.$$

If, thus, $\omega(x, y)$ is an integral of (3),

$$A(\omega) = X\frac{\partial \omega}{\partial x} + Y\frac{\partial \omega}{\partial y} = 0\,.$$

Since (3) admits, by hypothesis, of Uf,

$$U(\omega) = \Omega(\omega)\,.$$

We shall write, as usual,

$$U(Af) - A(Uf) = \{U(X) - A(\xi)\}\frac{\partial f}{\partial x} + \{U(Y) - A(\eta)\}\frac{\partial f}{\partial y}\,;$$

and if we put $f \equiv \omega$ in this identity, remembering that

$$U(A(\omega)) \equiv U(0) \equiv 0\,,$$

$$A(U(\omega)) \equiv A(\Omega(\omega)) \equiv \frac{d\Omega}{d\omega}\,.\,A(\omega) \equiv 0\,,$$

we find

$$\{U(X) - A(\xi)\}\frac{\partial\omega}{\partial x} + \{U(Y) - A(\eta)\}\frac{\partial\omega}{\partial y} \equiv 0\,.$$

Also, since

$$X\frac{\partial\omega}{\partial x} + Y\frac{\partial\omega}{\partial y} \equiv 0\,;$$

from the last two identities

$$\frac{U(X) - A(\xi)}{X} \equiv \frac{U(Y) - A(\eta)}{Y}\,.$$

If we call the value of these fractions $\lambda(x, y)$, this identity gives

$$U(X) - A(\xi) \equiv \lambda\,.\,X, \quad U(Y) - A(\eta) \equiv \lambda\,.\,Y\,.$$

Hence, for all values of f,

$$\{U(X) - A(\xi)\}\frac{\partial f}{\partial x} + \{U(Y) - A(\eta)\}\frac{\partial f}{\partial y} \equiv \lambda(x, y)\left\{X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y}\right\},$$

or, as it may be written,

$$U(Af) - A(Uf) \equiv \lambda(x, y)\,Af. \tag{4}$$

This is the condition that the differential equation (3) should admit of Uf. If, conversely, a condition (4) holds, the differential equation (3) must admit of Uf.

For, if ω be an integral of (3), $A(\omega) \equiv U(A(\omega)) \equiv 0$; or, from (4):

$$A(U(\omega)) \equiv \xi\frac{\partial U(\omega)}{\partial x} + \eta\frac{\partial U(\omega)}{\partial y} \equiv 0\,.$$

The last equation shows that $U(\omega)$ is a function of ω alone. Hence the integral curves of (3), that is, (3) itself will admit of Uf, if the condition (4) holds. Hence,

The differential equation

$$X(x, y)\,dy - Y(x, y)\,dx = 0$$

will admit then and only then of the G_1, Uf, when

$$U(Af) - A(Uf) \equiv \lambda(x, y)\,Af\,,$$

where

$$Af \equiv X(x, y)\frac{\partial f}{\partial x} + Y(x, y)\frac{\partial f}{\partial y}.$$

Let us illustrate this criterion by an example. As we know from geometrical considerations, the family of ∞^1 circles with equal radii,

$$(x - a)^2 + y^2 - r^2 = 0\,,$$

will admit of a translation along the x-axis. If we form the differential equation of these circles by the usual method, we find

$$ydy + \sqrt{r^2 - y^2}\,dx = 0\,.$$

In this case, therefore,

$$Af \equiv y\frac{\partial f}{\partial x} - \sqrt{r^2 - y^2}\frac{\partial f}{\partial y}.$$

The infinitesimal translation along the x-axis has the form

$$Uf \equiv \frac{\partial f}{\partial x}.$$

Hence, may be verified at once that

$$U(Af) - A(Uf) \equiv 0\,;$$

that is, since λ can be zero in (4), the criterion holds.

If the differential equation (3) happens to admit of *two* known infinitesimal transformations,

$$U_1 f \equiv \xi_1\frac{\partial f}{\partial x} + \eta_1\frac{\partial f}{\partial y}, \qquad U_2 f \equiv \xi_2\frac{\partial f}{\partial x} + \eta_2\frac{\partial f}{\partial y},$$

of which neither is *trivial*, then *two* integrating factors of (3) are known,

$$U_1 \equiv \frac{1}{X\eta_1 - Y\xi_1}, \qquad U_2 \equiv \frac{1}{X\eta_2 - Y\xi_2}.$$

It is a well-known theorem of the ordinary text-books on differential equations, that if U_1 and U_2 are two integrating factors of (3), then *the ratio* $U_1 : U_2$ *is either an integral of* (3), *or it is a constant.* When, therefore, (3) admits of *two* known infinitesimal transformations, one can very often find the integral of (3), without even a quadrature, by mere algebraic operations.

As an example, it can be easily verified that the differential equation

$$dy - (x - \sqrt{x^2 - 2y})\,dx = 0\,,$$

admits of

$$U_1 f \equiv \frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}, \qquad U_2 f \equiv x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y}.$$

Therefore the quotient $U_1 : U_2$ has the form

$$\frac{2y - (x - \sqrt{x^2 - 2y})\,x}{x - (x - \sqrt{x^2 - 2y})} = x - \sqrt{x^2 - 2y}\,,$$

and this is the integral of the above differential equation.

Our next object will be to show how to find all families of ∞^1 curves,—that is, all differential equations of the first order,—in the plane, which are invariant under a given G_1, Uf.

If the ∞^1 finite transformations of the G_1 be performed upon any curve in the plane which is not a *locus* of the G_1, a family of ∞^1 new curves will be obtained. Since the ∞^1 transformations form a G_1, a little reflection will show that this family of ∞^1 curves must, as a whole, be invariant, and no curve of the family can be a *locus* of the G_1.

If

$$\Omega(x_1, y_1) = 0$$

be the curve (taken for convenience in the variables x_1, y_1) upon which the ∞^1 transformations

$$x_1 = \varphi(x, y, a)\,, \quad y_1 = \psi(x, y, a)\,,$$

of the G_1 are performed, then the resulting invariant family will have the form

$$\Omega\big(\varphi(x, y, a)\,,\ \psi(x, y, a)\big) = 0\,.$$

To find the general form of the invariant differential equation of the first order, it is only necessary to eliminate the parameter a from the equations

$$\Omega(\varphi, \psi) = 0\,, \quad d\Omega(\varphi, \psi) = 0\,. \tag{5}$$

The differential equation of the ∞^1 *loci* of the G_1 will not be included in the general form of the invariant differential equation. Since, however, every G_1, according to the rule for integrating factors, is *trivial* as regards the differential equation of its own *loci*, it is a matter of no importance to us that we cannot obtain the differential equation of the *loci* from (5). It would be easy to show that when the equations to the finite transformations of a G_1 are given, the differential equation to the loci of the G_1, and its integral, can be found by processes which involve only differentiation and algebraic operations. In fact, this integral is the function which on a former occasion was called the *Invariant* of the G_1.*

The differential equations of the first order which are integrable by the methods of the ordinary text-books, are all such classes of differential equations as admit of certain G_1, as we shall show by the following elementary examples:—

* See Annals, Vol. VIII, No. 4, "Transformation Groups."

1°. Suppose we wish to find all differential equations of the first order, which are invariant under the G_1 of all translations along the x-axis.

The finite transformations of the G_1 have the form

$$x_1 = x + a\,, \quad y_1 = y\,, \quad a = \text{const.} \tag{6}$$

In order to find all invariant families of ∞^1 curves, we must perform (6) upon some curve in the plane. If the equation to this curve does not contain x, it will evidently be a *locus* of the G_1; and the equation to the *loci* can be written

$$y = \text{const.},$$

with the invariant differential equation

$$dy = 0\,.$$

On the other hand, if the equation to the curve with which we begin really contains x, it may be written in the form

$$x - \varphi(y) = 0\,.$$

By means of (6), this curve is transformed into the ∞^1 curves,

$$x - \varphi(y) = \text{const.}$$

The corresponding differential equation of the first order is

$$1 - \varphi'(y)\,.\,y' = 0\,. \tag{7}$$

Hence, *all differential equations of the first order, which are free of x, admit of the G_1 of all translations along the x-axis.*

The form (7), as is obvious, does not include the differential equation to the *loci*.

The infinitesimal transformation of the G_1 has the form

$$Uf \equiv \frac{\partial f}{\partial x}\,.$$

If (7) be written in the form

$$F(y)\,dy - dx = 0\,,$$

the rule for an integrating factor gives, in this case, 1. This is, of course, as it should be, since the left hand side of the last equation is already a complete differential.

2°. As a second example, let us find the differential equations of the first order which are invariant under the G_1 of the *affine* transformations in the plane.

The finite transformations of this G_1 have the form

$$x_1 = ax\,, \quad y_1 = y\,,$$

and the infinitesimal transformation is $x\,\dfrac{\partial f}{\partial x}$.

If the equation of the curve, upon which the ∞^1 finite transformations of the G_1 are performed, be free of x, it may be taken in the form

$$y = \text{const.}$$

If the constant in this equation be given ∞^1 different values, we evidently obtain the family of ∞^1 *loci* of the G_1, with the invariant differential equation

$$dy = 0 .$$

If the original curve contains x, it may be given the form

$$x - \varphi(y) = 0 .$$

Here φ cannot be zero; for every point on the line $x = 0$ is absolutely invariant. By the transformations the ∞^1 curves

$$\frac{x}{a} - \varphi(y) = 0$$

are obtained, or

$$\frac{\varphi(y)}{x} = \text{const.},$$

with the invariant differential equation

$$x\varphi'(y)\,dy - \varphi(y)\,dx = 0 .$$

If φ be a constant, this form gives the invariant differential equation

$$dx = 0 .$$

If φ be an arbitrary function which really contains y, the general differential equation of the first order which is invariant under the G_1 of affine transformations may evidently be written in the form

$$xdy - F(y)\,dx = 0 .$$

The rule gives the obvious integrating factor

$$\frac{1}{xF(y)},$$

for this differential equation.

3°. For a third example, let us take the G_1 of similitudinous transformations

$$x_1 = ax , \quad y_1 = ay , \tag{8}$$

with the infinitesimal transformation

$$Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} .$$

We know that the loci of the G_1 are the lines

$$\frac{y}{x} = \text{const.},$$

and we find thus the invariant differential equation

$$y' = \frac{x}{y}.$$

Let us perform all of the transformations (8) on a curve of the form

$$x - \varphi\left[\frac{y}{x}\right] = 0.$$

This curve must not be a *locus* of the G_1, and hence φ is not zero. If φ be a constant, we find the invariant differential equation

$$dx = 0.$$

If φ really contains $\frac{y}{x}$, we find, in the usual manner, the invariant family

$$ax - \varphi\left[\frac{y}{x}\right] = 0,$$

with the differential equation

$$\varphi\left[\frac{y}{x}\right] dx - \varphi'\left[\frac{y}{x}\right]\left[dy - \frac{y}{x}\, dx\right] = 0.$$

This may be written in the form

$$y' = F\left[\frac{y}{x}\right], \tag{9}$$

and we see that *the general homogeneous differential equation of the first order admits of the G_1 of the similitudinous transformations.*

$F\left[\frac{y}{x}\right]$ in (9) can have any value except $\frac{y}{x}$, (since φ is not 0). Hence the rule gives the integrating factor

$$M = \frac{1}{y - xF\left[\frac{y}{x}\right]}.$$

To shorten the work in this example the curve

$$x - \varphi\left[\frac{y}{x}\right] = 0$$

is chosen in a somewhat artificial manner; but it may be readily verified that

the same results would have been arrived at if we had started with a curve of the form

$$x - \varphi(y) = 0 .$$

4°. If we make use of polar coordinates, the G_1 of rotations around the origin is given by the equations

$$r_1 = r , \quad \varphi_1 = \varphi + a , \tag{10}$$

where a is the amplitude of the rotation. We may choose

$$\omega(r_1, \varphi_1) = 0 \tag{11}$$

as the curve to begin with; and by (10) we obtain from (11) the invariant family

$$\omega(r, \varphi + a) = 0 . \tag{12}$$

If ω in (11) does not contain φ_1, it is evident that (12) will not contain $\varphi + a$; in other words,

$$r = \text{const.},$$

or, in rectangular coordinates,

$$x^2 + y^2 = \text{const.}$$

is the invariant family of *loci*. This gives the invariant differential equation

$$xdx + ydy = 0 .$$

If, however, (11) really contains φ_1, (12) may be solved in the form

$$\varphi - f(r) = \text{const.}$$

That is, in rectangular coordinates, the invariant family of curves will have the form:

$$\tan^{-1}\frac{y}{x} - \Phi(x^2 + y^2) = \text{const.}$$

This gives the differential equation

$$\frac{xy' - y}{x + yy'} = F(x^2 + y^2) . \tag{13}$$

An infinitesimal rotation around the origin has the form

$$Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} .$$

Hence, an integrating factor of (13) must have the form

$$M = \frac{1}{(x - yF)x + (y + xF)y} = \frac{1}{x^2 + y^2} .$$

5°. It may be easily verified that the equations

$$x_1 = x , \quad y_1 = y + a . \varphi(x) , \tag{14}$$

define a G_1. The family of the ∞^1 invariant *loci* evidently are defined by the invariant differential equation

$$dx = 0\,.$$

Let us perform the transformations (14) upon a curve of the form

$$y - \psi(x) = 0\,.$$

This gives the invariant family

$$y - a\,.\,\varphi(x) - \psi(x) = 0\,,$$

with the differential equation

$$dy - \left\{ \frac{y - \psi(x)}{\varphi(x)}\,.\,\varphi'(x) + \psi'(x) \right\} dx = 0\,.$$

If, now, we write

$$\Phi(x) = \frac{\varphi'(x)}{\varphi(x)}\,,$$

$$\Psi(x) = \psi'(x) - \psi(x)\,.\,\frac{\varphi'(x)}{\varphi(x)}\,,$$

the differential equation becomes

$$y' - \Phi(x)\,.\,y - \Psi(x) = 0\,. \tag{15}$$

This is the general *linear* differential equation of the first order. We have

$$\varphi(x) = e^{\int \Phi(x)dx},$$

and hence (15) admits of the G_1

$$x_1 = y\,, \quad y_1 = y + a\,.\,e^{\int \Phi(x)\,dx}\,.$$

The infinitesimal transformation of this G_1 has the form

$$e^{\int \Phi(x)\,dx}\,.\,\frac{\partial f}{\partial y}\,;$$

and we find as integrating factor of the general linear differential equation of the first order (15)

$$M = \frac{1}{e^{\int \Phi(x)\,dx}}\,.$$

These examples, which could be multiplied indefinitely, serve to show that the classes of differential equations of the first order which were integrated by the old methods can all be defined as admitting of the ∞^1 transformations of a certain G_1, and the integrating factor sought can be immediately given.

As we know, each infinitesimal transformation in the plane generates a G_1, and each G_1 has its invariant differential equations of the first order, which can be found by the above methods. All such differential equations are, of course, immediately integrable, and it would be both interesting and useful to tabulate the simplest of these classes of integrable differential equations.

GEOMETRICAL MULTIPLICATION OF SURFACES.

By Dr. A. S. Chessin, Baltimore, Md.

Among the expressions which have to be transformed to a new system of coordinates in Analytical Mechanics there is one of frequent occurrence, namely the one of the form

$$(y_1z_2 - y_2z_1)(y_4z_3 - y_3z_4) + (z_1x_2 - z_2x_1)(z_4x_3 - z_3x_4) + (x_1y_2 - y_1x_2)(y_3x_4 - x_3y_4) \quad (1)$$

where x_i, y_i, z_i are the rectangular coordinates of a point i. I will show that this expression is *invariant* in regard to a rotation of the system of axes of coordinates.

I will call the expression $S_1S_2 \cos(\widehat{S_1S_2})$ the *geometrical product* of the two plane surfaces S_1 and S_2, analogously to the expression "geometrical product" of two straight lines l_1 and l_2 for $l_1l_2 \cos(\widehat{l_1l_2})$. Let, then, first Δ_1 and Δ_2 be the surfaces of two triangles. We shall have the following

Theorem: The geometrical product of two triangles is equal to the sum of the products of their projections on the planes of coordinates; i. e.

$$\Delta_1\Delta_2 \cos(\Delta_1\Delta_2) = \Delta_{1x}\Delta_{2x} + \Delta_{1y}\Delta_{2y} + \Delta_{1z}\Delta_{2z}\,. \quad (2)$$

It is obvious, that a translation of the system parallel to itself does not alter formula (2); we may therefore suppose, that the two triangles have one vertex in common and that this point is taken as the origin 0 of the axes of coordinates. Let then Δ_1 be formed by the points 0, 1, 2; and Δ_2 by the points 0, 3, 4. We shall have:

$$2\Delta_{1x} = y_1z_2 - z_1y_2\,, \qquad 2\Delta_{2x} = y_4z_3 - y_3z_4\,,$$

$$2\Delta_{1y} = z_1x_2 - x_1z_2\,, \qquad 2\Delta_{2y} = z_4x_3 - z_3x_4\,,$$

$$2\Delta_{1z} = x_1y_2 - y_1x_2\,, \qquad 2\Delta_{2z} = x_4y_3 - x_3y_4\,,$$

by means of which formulæ (1) can be transformed into

$$4(\Delta_{1x}\Delta_{2x} + \Delta_{1y}\Delta_{2y} + \Delta_{1z}\Delta_{2z})\,. \quad (3)$$

On the other hand (1) can be transformed into

$$(x_2x_3 + y_2y_3 + z_2z_3)(x_1x_4 + y_1y_4 + z_1z_4) - (x_2x_4 + y_2y_4 + z_2z_4)(x_3x_1 + y_3y_1 + z_3z_1)$$

by a well known formula due to Euler, or into this equivalent form,

$$r_1r_2r_3r_4\left[\cos(\widehat{14})\cos(\widehat{23}) - \cos(\widehat{24})\cos(\widehat{13})\right], \quad (4)$$

where r_i is the distance of the point i from the origin; $\cos(\widehat{ij})$ is the cos of the angle formed by the lines $0i$ and $0j$. We have, therefore,

$$4(\varDelta_{1x}\varDelta_{2x}+\varDelta_{1y}\varDelta_{2y}+\varDelta_{1z}\varDelta_{2z})=r_1r_2r_3r_4\left[\cos(\widehat{14})\cos(\widehat{23})-\cos(\widehat{24})\cos(\widehat{13})\right]. \quad (5)$$

Let us now draw a sphere with the radius unity and the centre in 0. Let, further, (1), (2), (3), (4) be the points of intersection of this sphere with the lines 01, 02, 03, 04. Finally, let (5) be the point of intersection of the great circles $\widehat{12}$ and $\widehat{34}$, and A be the angle between them, i. e. $A=\angle\,\widehat{\varDelta_1\varDelta_2}$. This angle is perfectly determined, if we agree to call positive normals such that a rotation about them will carry a certain point i towards a certain point j, as well in the plane of the triangles as in their projections. Then we shall have the following relations:

$$\cos(\widehat{31})=\cos(\widehat{35})\cos(\widehat{21}+\widehat{25})+\sin(\widehat{35})\sin(\widehat{21}+\widehat{25})\cos A\,,$$

$$\cos(\widehat{24})=\cos(\widehat{25})\cos(\widehat{34}+\widehat{35})+\sin(\widehat{25})\sin(\widehat{34}+\widehat{35})\cos A\,,$$

$$\cos(\widehat{14})=\cos(\widehat{21}+\widehat{25})\cos(\widehat{34}+\widehat{35})+\sin(\widehat{21}+\widehat{25})\sin(\widehat{34}+\widehat{35})\cos A\,,$$

$$\cos(\widehat{23})=\cos(\widehat{25})\cos(\widehat{35})+\sin(\widehat{25})\sin(\widehat{35})\cos A\,;$$

from which we find that

$$\cos(\widehat{14})\cos(\widehat{23})-\cos(\widehat{13})\cos(\widehat{24})=\sin(\widehat{12})\sin(\widehat{34})\cos A\,.$$

Hence

$$4(\varDelta_{1x}\varDelta_{2x}+\varDelta_{1y}\varDelta_{2y}+\varDelta_{1z}\varDelta_{2z})=r_1r_2r_3r_4\sin(\widehat{12})\sin(\widehat{34})\cos A\,;$$

or as we have

$$2\varDelta_1=r_1r_2\sin(\widehat{12})\,,\qquad 2\varDelta_2=r_3r_4\sin(\widehat{34})\,,$$

we finally obtain

$$\varDelta_{1x}\varDelta_{2x}+\varDelta_{1y}\varDelta_{2y}+\varDelta_{1z}\varDelta_{2z}=\varDelta_1\varDelta_2\cos(\widehat{\varDelta_1\varDelta_2})$$

i. e. formula (2), which proves the theorem. It follows from the above, that a translation and rotation of the system of rectangular axes of coordinates into a new one (ξ, η, ζ) with the origin at the point (a, b, c) will change the expression (1) into the following:

$$\begin{aligned}&[(\eta_1+b)(\zeta_2+c)-(\eta_2+b)(\zeta_1+c)]\,[(\eta_4+b)(\zeta_3+c)-(\eta_3+b)(\zeta_4+c)]\\ +\,&[(\zeta_1+c)(\xi_2+a)-(\zeta_2+c)(\xi_1+a)]\,[(\zeta_4+c)(\xi_3+a)-(\zeta_3+c)(\xi_4+a)]\\ +\,&[(\xi_1+a)(\eta_2+b)-(\xi_2+a)(\eta_1+b)]\,[(\xi_4+a)(\eta_3+b)-(\xi_3+a)(\eta_4+b)].\end{aligned}$$

It is well to notice, that we are not restricted to lines purely, because the theorem is true whatever quantities are represented by the lines; thus, for instance, they may represent forces.

The above theorem can be generalized without further details, if the fact that any plane surface can be regarded as the limit of a polygonal surface, be taken into consideration; because every polygon may be divided into triangles, and then the above theorem applied. Hence:

The geometrical product of two plane surfaces S_1 and S_2 is equal to the sum of the products of their projections on the planes of coordinates, i. e.

$$S_1 S_2 \cos(\widehat{S_1 S_2}) = S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z} .$$

To conclude, I will give formula (2) in form of determinants:

$$4 J_1 J_2 \cos(\widehat{J_1 J_2}) = \left\| \begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix} \right\| \cdot \left\| \begin{matrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{matrix} \right\|$$

$$= \begin{vmatrix} x_1 x_4 + y_1 y_4 + z_1 z_4, & x_1 x_3 + y_1 y_3 + z_1 z_3 \\ x_2 x_4 + y_2 y_4 + z_2 z_4, & x_2 x_3 + y_2 y_3 + z_2 z_3 \end{vmatrix} = - \begin{vmatrix} x_1 & x_2 & -1 & 0 & 0 \\ y_1 & y_2 & 0 & -1 & 0 \\ z_1 & z_2 & 0 & 0 & -1 \\ 0 & 0 & x_3 & y_3 & z_3 \\ 0 & 0 & x_4 & y_4 & z_4 \end{vmatrix} .$$

In the special case, when the triangles are parallel and equal, this formula becomes:

$$4 J^2 = \left\| \begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix} \right\|^2 \quad \text{or} \quad 2 J = \left\| \begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix} \right\|$$

a well known formula.

Johns Hopkins University, *October* 30, 1894.

ON THE INSCRIPTION OF REGULAR POLYGONS.

By Mr. Leonard E. Dickson, Chicago, Ill.

Heretofore* it has been my aim to treat this geometric subject without the use of the customary complex imaginary. It is now my purpose to avoid the use of trigonometry also, basing my treatment solely upon algebraic and geometric principles.

The following two theorems, which are stated and proven geometrically in the last edition of Catalan's *Géométrie*, form the basis of the treatment:

Suppose a circle of unit radius divided at $A, A_1, A_2, A_3, \ldots, A_p, \ldots$ into $2p + 1$ equal parts and the diameter AO drawn. Join O to $A_1, A_2, \ldots, A_p$. Let OA_r and OA_s (or briefly A_r and A_s) denote any two of these chords.

Theorem I. If $r + s \leqq p$, $\quad A_r \,.\, A_s = A_{s-r} + A_{s+r}$. (1)

Theorem II. If $r + s > p$, $\quad A_r \,.\, A_s = A_{s-r} - A_{n-(s+r)}$. (2)

These follow at sight in their trigonometric form. Thus, (1) gives

$$2 \cos \frac{r\pi}{n} \,.\, \cos \frac{s\pi}{n} = \cos \frac{(s-r)\pi}{n} + \cos \frac{(s+r)\pi}{n},$$

where $n = 2p + 1$.

Corollary. $A_s^2 = 2 + A_{2s}$ if $2s \leqq p$; but $= 2 - A_{n-2s}$ if $2s > p$. (3)

The following theorem is fundamental:

$$A_1 - A_2 + A_3 - A_4 + A_5 - \ldots - (-1)^p A_p = 1. \tag{4}$$

Proof. Suppose $A_1 - A_2 + A_3 - \ldots \pm A_p = x$.

Then
$$\begin{aligned} xA_1 &= A_1 (A_1 - A_2 + A_3 - \ldots \pm A_{p-2} \mp A_{p-1} \pm A_p) \\ &= 2 + A_2 - A_1 - A_3 + A_2 + A_4 - A_3 - A_5 + \ldots \\ &\quad \pm A_{p-3} \pm A_{p-1} \mp A_{p-2} \mp A_p \pm A_{p-1} \mp A_p, \end{aligned}$$

by applying (1) at every step of the multiplication except the first and last, when we apply (3) and (2) respectively.

$$\begin{aligned} \therefore\ xA_1 &= 2 - A_1 + 2 (A_2 - A_3 + A_4 - A_5 + \ldots \mp A_{p-2} \pm A_{p-1} \mp A_p) \\ &= 2 + A_1 - 2 (A_1 - A_2 + A_3 - A_4 + \ldots \pm A_{p-2} \mp A_{p-1} \pm A_p) \\ &= 2 + A_1 - 2x. \end{aligned}$$

$$\therefore\ (x - 1)(2 + A_1) = 0.$$

* An elementary sketch of my method appears in a series of articles (beginning Sept., 1894) in the American Mathematical Monthly.

But A_1 is not equal to -2. $\therefore x = 1$.

Compare the following direct trigonometric proof:

In the identity,

$$\cos a + \cos 3a + \cos 5a + \ldots + \cos(2p-1)a = \frac{\sin 2pa}{\sin a},$$

let

$$a = \frac{\pi}{2p+1}.$$

Then

$$a = \angle AOA_1,\ 3a = \angle AOA_3,\ \text{etc.}$$

Now

$$\cos(2p-1)a = \cos\frac{2p-1}{2p+1}\pi = -\cos\frac{2\pi}{2p+1};$$

$$\cos(2p-3)a = -\cos\frac{4\pi}{2p+1};\ \text{etc.}$$

Also

$$\sin 2pa = \sin\frac{2p\pi}{2p+1} = \sin\frac{\pi}{2p+1}.$$

$$\therefore\ 2\cos\frac{\pi}{2p+1} - 2\cos\frac{2\pi}{2p+1} + 2\cos\frac{3\pi}{2p+1} - 2\cos\frac{4\pi}{2p+1} + \ldots$$

$$\pm 2\cos\frac{p\pi}{2p+1} = 1.$$

To construct the equation whose roots are $A_1, -A_2, A_3, -A_4, \ldots, -(-1)^p A_p$.

$\Sigma A_i = 1$, where the plus sign is concealed in the root if i be odd and the minus sign if i be even. We may form the symmetric functions of the p roots, as follows:

$$(\Sigma A_i)^2 = \Sigma A_i^2 + 2\Sigma A_i A_j.$$

But

$$A_i^2 = 2 + A_{2i}.$$

Hence

$$\Sigma A_i^2 = 2p + \Sigma A_{2i} = 2p - \Sigma A_i = 2p - 1.$$

$$\therefore\ \Sigma A_i A_j = -(p-1).$$

$$(\Sigma A_i)^3 = \Sigma A_i^3 + 3\Sigma A_i^2 A_j + 6\Sigma A_i A_j A_k.$$

But

$$A_i^3 = A_i(2 + A_{2i}) = 3A_i + A_{3i}.$$

$$\therefore\ \Sigma A_i^3 = 3\Sigma A_i + \Sigma A_{3i} = 4\Sigma A_i = 4.$$

$$\Sigma A_i^2 A_j = \Sigma A_i^2 \Sigma A_j - \Sigma A_i^3 = (2p-1) - 4.$$

$$\therefore\ \Sigma A_i A_j A_k = -(p-2).$$

Whether we proceed by this method or the preferable ones below, it is necessary to find ΣA_i^m, denoted by s_m. We have found that

$$s_1 = 1, \quad s_2 = 2p - 1 = n - 2, \quad s_3 = 4.$$

$$A_i^4 = A_i(3A_i + A_{3i}) = 6 + 4A_{2i} + A_{4i}; \quad \therefore s_4 = 6p - 5 = 3n - 8.$$

$$A_i^5 = A_i(6 + 4A_{2i} + A_{4i}) = 10A_i + 5A_{3i} + A_{5i}; \quad \therefore s_5 = 16.$$

$$A_i^6 = 20 + 15A_{2i} + 6A_{4i} + A_{6i}; \quad \therefore s_6 = 20p - 22 = 10n - 32.$$

Similarly, $s_7 = 64$, $s_8 = 35n - 128$, $s_9 = 256$, $s_{10} = 126n - 512$.

Theorem. *For the sum of like odd powers,* $s_{2k+1} = 2^{2k}$. (5)

$A_i^{2k-2} = (2 + A_{2i})^{k-1}$ gives on expansion a numerical term and terms linear in $A_{2i}, A_{4i}, A_{6i}, \ldots, A_{(2k-2)i}$, but no terms whose subscript is an *odd* multiple of i. Hence A_i^{2k-1} will contain no numerical term and no term whose subscript is an *even* multiple of i. We may therefore write

$$A_i^{2k-1} = aA_i + bA_{3i} + cA_{5i} + dA_{7i} + \ldots + gA_{(2k-1)i}.$$

Then $A_i^{2k+1} = (2 + A_{2i})\,A_i^{2k-1}$

$$= 2aA_i + 2bA_{3i} + 2cA_{5i} + \ldots + 2gA_{(2k-1)i}$$

$$+ a(A_i + A_{3i}) + b(A_i + A_{5i}) + c(A_{3i} + A_{7i}) + \ldots + g(A_{(2k-3)i} + A_{(2k+1)i})$$

$$= (3a + b)\,A_i + (a + 2b + c)\,A_{3i} + (b + 2c + d)\,A_{5i} + (c + 2d + e)\,A_{7i} + \ldots.$$

$$\therefore s_{2k-1} = a + b + c + d + \ldots + g, \qquad s_{2k+1} = 4(a + b + c + d + \ldots + g).$$

$$\therefore s_{2k+1} = 4s_{2k-1} = 2^4 s_{2k-3} = 2^6 s_{2k-5} = \ldots = 2^{2k} s_1 = 2^{2k}.$$

Theorem. *For the sum of like even powers of the roots,* $s_{2k} = a_{2k-1}n - 2^{2k-1}$, *where* a_{2k-1} *is the coefficient of* A_i *in the expansion of* A_i^{2k-1}. (6)

$$A_i^{2k} = A_i(A_i^{2k-1})$$

$$= a(2 + A_{2i}) + b(A_{2i} + A_{4i}) + c(A_{4i} + A_{6i}) + \ldots + g(A_{(2k-2)i} + A_{2ki})$$

$$= 2a + (a + b)A_{2i} + (b + c)A_{4i} + (c + d)\,A_{6i} + \ldots + (f + g)A_{(2k-2)i} + gA_{2ki}.$$

$$\therefore \quad s_{2k} = 2ap - (a + b) - (b + c) - (c + d) \ldots - (f + g) - g$$

$$= 2ap + a - 2(a + b + c + d + \ldots + g).$$

But $$(a + b + c + d + \ldots + g) = s_{2k-1} = 2^{2k-2};$$

$$\therefore \quad s_{2k} = 2ap + a - 2^{2k-1} = an - 2^{2k-1}.$$

Another proof of (6), which is needed below, is as follows:

Write $$A_i^{2k} = a + BA_{2i} + CA_{4i} + DA_{6i} + \dots + RA_{2ki}.$$

Then $$s_{2k} = ap - (B + C + D + \dots + R).$$

$$A_i^{2k+1} = aA_i + B(A_i + A_{3i}) + C(A_{3i} + A_{5i}) + \dots$$
$$= (a + B)A_i + (B + C)A_{3i} + (C + D)A_{5i} + \dots + RA_{(2k+1)i}.$$

$$\therefore\ s_{2k+1} = a + 2(B + C + D + \dots + R) = a + 2ap - 2s_{2k} = na - 2s_{2k}.$$

But $a_{2k} = 2a_{2k-1}$, and $s_{2k+1} = 2^{2k}$; $\therefore\ s_{2k} = a_{2k-1}n - 2^{2k-1}$.

We may state this formula thus:

$$2s_{2k} = na_{2k} - 2^{2k}. \tag{7}$$

An interesting relation is derived as follows:

$$A_i^{2k+2} = (2 + A_{2i})A_i^{2k}$$
$$= 2a + 2BA_{2i} + 2CA_{4i} + \dots + aA_{2i} + B(2 + A_{4i}) + C(A_{2i} + A_{6i}) + \dots$$
$$= 2(a + B) + (a + 2B + C)A_{2i} + (B + 2C + D)A_{4i}$$
$$+ (C + 2D + E)A_{6i} + \dots + (Q + 2R)A_{2ki} + RA_{(2k+2)i}.$$

$$\therefore\ s_{2k+2} = 2(a + B)p - (a + 2B + C) - (B + 2C + D) - (C + 2D + E) - \dots$$
$$- (Q + 2R) - R$$
$$= 2(a + B)p - a + B - 4(B + C + D + \dots + Q + R)$$
$$= 2(a + B)p - a + B + 4s_{2k} - 4ap$$
$$= -2ap + 2Bp - a + B + 4s_{2k} = n(B - a) + 4s_{2k}.$$

$$\therefore\ s_{2k+2} - 4s_{2k} = n(B_{2k} - a_{2k}). \tag{8}$$

To obtain the value of a_{2k-1}, break a Pascal Triangle along the diagonal indicated in Table 1 by the heavy figures; discard the part to the right, and turn the part to the left over. Thus Table 2, aside from this diagonal, gives the coefficients in the linear expansion of A_i^m. The reason why these coefficients obey the law of Pascal's Triangle $C_{p+1}^{q+1} = C_p^{q+1} + C_p^q$ is found in the proofs of (6) and (7).

C	0	1	2	3	4	5	6	7	8	9	... q
0	1										
1	1	**1**									
2	1	2	1								
3	1	3	**3**	1							
4	1	4	6	4	1						
5	1	5	10	**10**	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	**35**	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	**126**	84	36	9	1	
...											
p											

TABLE 1.

5	4	3	2	1	0	C
					1	0
				1	1	1
				2	1	2
			3	3	1	3
			6	4	1	4
		10	10	5	1	5
		20	15	6	1	6
	35	35	21	7	1	7
	70	56	28	8	1	8
126	126	84	36	9	1	9
252	210	120	45	10	1	10

TABLE 2.

Since the mth term of the series 1, 3, 10, 35, 126, ... (denoted by a_{2m-1}) is at the intersection of column m with row $(2m-1)$, it is

$$(2m-1)(2m-2)(2m-3)\ldots(2m-m)/(1.2.3\ldots m);$$

$$\therefore\quad a_{2m-1}=\frac{(2m-1)(2m-2)\ldots(m+1)}{1.2.3\ldots(m-1)}=\frac{(2m)!}{2.(m!)^2}. \tag{9}$$

Thus,

$$a_{2m-3} = \frac{(2m-3)(2m-4)\dots(m)}{1.2\dots(m-2)}; \quad \therefore\ ma_{2m-1} = 2(2m-1)\,a_{2m-3}.$$

It follows that

$$a_{2m-1} = \frac{(2m-1)(2m-3)(2m-5)\dots 3}{1.2.3\dots m} \,.\, 2^{m-1}.$$

We have thus proven that

$$s_{2k+1} = 2^{2k}; \quad s_{2k} = \frac{(2k-1)(2k-2)\dots(k+1)}{1.2.3\dots(k-1)}\,n - 2^{2k-1}. \tag{10}$$

The coefficients of n are 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, etc.

The general coefficient C_m in the equation sought, expressed in terms of the sums of like powers of the roots, is

$$C_m = \Sigma \frac{(-1)^{t_1+t_2+\dots+t_m}\, s_1^{t_1} s_2^{t_2}\dots s_m^{t_m}}{\pi(t_1+1)\,\pi(t_2+1)\dots\pi(t_m+1)\,2^{t_2}3^{t_3}\dots m^{t_m}}, \tag{11}$$

where $t_1, t_2, \dots, t_m$ take all positive values, including zero, which satisfy the equation

$$t_1 + 2t_2 + 3t_3 + \dots + mt_m = m, \tag{12}$$

and $\pi(t_i+1) = 1.2.3\dots t_i$, with the assumption $\pi(1) = 1$.

In our problem,

$$s_1^{t_1} s_2^{t_2}\dots s_m^{t_m} = (n-2)^{t_2}.\,2^{2t_3}.\,(3n-2^3)^{t_4}.\,2^{4t_5}\dots$$

$$= 2^{(2t_3+4t_5+6t_7+\dots+2kt_{2k+1}+\dots)}(n-2)^{t_2}(3n-2^3)^{t_4}(10n-2^5)^{t_6}\dots$$

$$\times\left[\frac{(2k-1)(2k-2)\dots(k+1)}{1.2.3\dots(k-1)}\,n - 2^{2k-1}\right]^{2k}\dots.$$

Thus,

$$C_3 = \frac{-2^2}{3} + \frac{n-2}{2} - \frac{1}{2.3} = \frac{n-5}{2} = (p-2)$$

$$C_4 = \frac{-(3n-8)}{4} + \frac{2^2}{3} - \frac{n-2}{4} + \frac{(n-2)^2}{2.4} + \frac{1}{2.3.4}$$

$$= \tfrac{1}{8}(n-5)(n-7) = \tfrac{1}{2}(p-2)(p-3).$$

We may determine these coefficients by the equation

$$(-1)^m m!\,C_m = \begin{vmatrix} s_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & 0 & 0 & \dots & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_m & s_{m-1} & s_{m-2} & s_{m-3} & \dots & \dots & \overset{m-1}{s_1} \end{vmatrix} \qquad (13)$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ n-2 & 1 & 2 & 0 & 0 & 0 & \dots \\ 2^2 & n-2 & 1 & 3 & 0 & 0 & \dots \\ 3n-2^3 & 2^2 & n-2 & 1 & 4 & 0 & \dots \\ 2^4 & 3n-2^3 & 2^2 & n-2 & 1 & 5 & \dots \\ 10n-2^5 & 2^4 & 3n-2^3 & 2^2 & n-2 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Multiplying each row by 2 and adding to the one below,

$$(-1)^m m!\,C_m = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ n & 3 & 2 & 0 & 0 & 0 & \dots \\ 2n & n & 5 & 3 & 0 & 0 & \dots \\ 3n & 2n & n & 7 & 4 & 0 & \dots \\ 6n & 3n & 2n & n & 9 & 5 & \dots \\ 10n & 6n & 3n & 2n & n & 11 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \qquad (13_a)$$

By either method we obtain the equation whose roots are the chords A_1, $-A_2$, A_3, $-A_4$, ..., $-(-1)^p A_p$ of the circle of unit radius, viz.,

$$x^p - x^{p-1} - (p-1)x^{p-2} + (p-2)x^{p-3} + \frac{(p-2)(p-3)}{1\,.\,2}x^{p-4}$$
$$- \frac{(p-3)(p-4)}{1\,.\,2}x^{p-5} - \dots$$
$$+ (-1)^m \frac{(p-m)(p-m-1)\dots(p-2m+1)}{1\,.\,2\,.\,3\dots m}x^{p-2m}$$
$$-(-1)^m \frac{(p-m-1)(p-m-2)\dots(p-2m)}{1\,.\,2\,.\,3\dots m}x^{p-2m-1} - \dots = 0. \qquad (14)$$

I will not enter into the decomposition of this equation. Its irreducibility in the domain of rational numbers follows from Eisenstein's Theorem.

For regular polygons of a composite number of sides $n = rm$, we have the fundamental theorem

$$A_s - A_{m-s} - A_{m+s} + A_{2m-s} + A_{2m+s} - \dots \\ + (-1)^{\frac{1}{2}(r-1)} (A_{\frac{1}{2}(r-1)m-s} + A_{\frac{1}{2}(r-1)m+s}) = 0 . \tag{15}$$

s being any integer $\leqq \frac{1}{2}(m-1)$.

PROOF: $A_m - A_{2m} + A_{3m} - A_{4m} + \dots - (-1)^{\frac{1}{2}(r-1)} A_{\frac{1}{2}(r-1)m} = 1$, since A_m, etc. are chords of the regular r-gon. Hence

$$A_s = A_s(A_m - A_{2m} + A_{3m} - \dots)$$
$$= A_{m-s} + A_{m+s} - A_{2m-s} - A_{2m+s} + \dots .$$

To find the equation whose roots are A_s, $-A_{m-s}$, $-A_{m+s}$, A_{2m-s}, A_{2m+s}; $\dots (-1)^{\frac{1}{2}(r-1)} A_{\frac{1}{2}(r-1)m\pm s}$, we proceed as in the case of a regular polygon of a prime number of sides. Evidently the coefficients in the expansion of $A^t_{km\pm s}$ are the same as those found for A^t_i. But in taking the sum of the r expressions for the like powers of the r roots, we have the sum of the roots $= 0$ now. Further, since n is a composite number, we may find in the expansion of $A^t_{km\pm s}$ chords which belong also to a regular polygon the number of whose sides is a divisor of n. In that case, in summing we obtain sub-groups of the chords instead of the entire groups (15).

Thus, for a regular polygon of 35 sides, take $m = 7$, $r = 5$;

$$\therefore \quad A_1 - A_6 - A_8 + A_{13} + A_{15} = 0 .$$

$$S_2 = A_1^2 + A_6^2 + A_8^2 + A_{13}^2 + A_{15}^2$$
$$= 10 + (A_2 + A_{12} + A_{16} - A_9 - A_5) = 10 \text{ by (15)}.$$

$$S_3 = A_1^3 - A_6^3 - A_8^3 + A_{13}^3 + A_{15}^3$$
$$= 3(A_1 - A_6 - A_8 + A_{13} + A_{15}) + (A_3 + A_{17} + A_{11} - A_4 - A_{10}) = 0 .$$

$$S_4 = \Sigma_i(6 + 4A_{2i} + A_{4i})$$
$$= 5.6 + 4(A_2 + A_{12} + A_{16} - A_9 - A_5) + (A_4 - A_{11} - A_3 - A_{17} + A_{10}) = 5.6.$$

But

$$S_5 = \Sigma_i(10A_i + 5A_{3i} + A_{5i})$$
$$= 10(A_1 - A_6 - A_8 + A_{13} + A_{15}) + 5(A_3 + A_{17} + A_{11} - A_4 - A_{10})$$
$$+ 5A_5 = 5A_5 .$$

Similarly,

$$S_7 = 35(A_1 - A_6 \ldots) + 21(A_3 + A_{17} \ldots) + 35A_5 + (2A_7 - 2A_{14} - 2)$$
$$= 35A_5,$$

since $A_7 - A_{14} = 1$, by (4), being chords of the pentagon.

For a regular 105-gon, take $m = 7$, $r = 15$. Then

$$A_1 - A_6 - A_8 + A_{13} + A_{15} - A_{20} - A_{22} + A_{27} + A_{29} - A_{34} - A_{36} + A_{41}$$
$$+ A_{43} - A_{48} - A_{50} = 0.$$

$$S_2 = 30 + A_2 - A_5 - A_9 + A_{12} + A_{16} - \ldots$$
$$= 30.$$

$$S_3 = 3(A_1 - A_6 - A_8 + A_{13} + \ldots) + 3(A_3 - A_{18} - A_{24} + A_{39} + A_{45}).$$

But applying (15) for $s = 3$, $m = 21$,

$$A_3 - A_{18} + A_{24} + A_{39} + A_{45} = 0. \quad \therefore S_3 = 0.$$

$$S_4 = 6.15 + 4(A_2 - A_5 - A_9 + A_{12} + A_{16} \ldots) - (A_3 - A_4 - A_{10} + A_{11} + \ldots)$$
$$= 6.15.$$

$$S_5 = 10(A_1 - A_6 - A_8 + \ldots) + 15(A_3 - A_{18} - A_{24} + A_{39} + A_{45})$$
$$+ 5(A_5 - A_{30} - A_{40}).$$

But $$A_s - A_{35-s} - A_{35+s} = 0; \quad \therefore S_5 = 0.$$

$$S_6 = 15.20 + 15(A_2 - A_5 - A_9 + \ldots) - 6(A_3 - A_4 - A_{10} + \ldots)$$
$$+ 3(A_6 + A_{36} + A_{48} - A_{27} - A_{15}) = 15.20.$$

If the groups occurring in the expansion of S are all of the form (15), and hence zero, it is necessary to examine in S_{k+2} only the last group, viz., that arising from $\Sigma_i A_{(k+2)i}$, where i takes the r values $s, m - s, m + s, 2m - s, \ldots$; for all the previous groups are the same as those in S_k.

Thus in S_7, $\Sigma A_{7i} = 2(A_7 - A_{14} + A_{21} - A_{28} + A_{35} - A_{42} + A_{49}) - 2 = 0$ by (4); in S_9, $\Sigma A_{9i} = 3(A_9 - A_{12} - A_{30} + A_{33} - A_{51}) = 0$ by (15), for $s = 9$, $m = 21$; in S_{11}, $\Sigma A_{11i} = A_{11} + A_{39} + A_{17} - A_{38} + A_{45} - A_{10} - A_{32} - A_{18} - A_4 - A_{46} - A_{24} + A_{31} - A_{52} + A_3 + A_{25} = 0$, by (15), for $s = 3$, $m = 7$; similarly $\Sigma A_{13i} = 0$. But in S_{15}, $\Sigma A_{15i} = 15A_{15}$.

$$\therefore S_7 = S_9 = S_{11} = S_{13} = 0; \quad S_{15} = 15A_{15}.$$

In forming the sum of like powers of the chords in the group (15), the common exponent being *prime* to n, we get, besides the numerical term, multiples of groups of the same form (15).

For the sum of like powers of the chords, the common exponent being $< r$, but not prime to n, we get, besides multiples of groups (15), multiples of sub-groups of the form

$$A_s - A_{km-s} - A_{km+s} + A_{2km-s} + A_{2km+s} - \ldots + (-1)^{\frac{1}{2}\left(\frac{r}{k}-1\right)} A_{\frac{1}{2}\left(\frac{r}{k}-1\right)m \pm s} = 0 .$$

Hence
$$S_{2l} = 2a_{2l-1}r , \tag{16}$$
in which a_{2l-1} is given by (9). But when this exponent $= r$, the last term ΣA_{rt} in the expansion of ΣA_t^r is

$$\begin{aligned} &A_{rs} - A_{rm-rs} - A_{rm+rs} + A_{2rm-rs} + A_{2rm+rs} - \ldots \\ &= A_{rs} + A_{rs} + A_{rs} + A_{rs} + A_{rs} + \ldots \\ &= rA_{rs} . \end{aligned}$$

$\therefore\ S_{2l+1} = 0$, if $2l + 1 < r$; while
$$S_r = rA_{rs} . \tag{17}$$

It follows from (11) that, in the equation sought, every coefficient k_{2l+1} (if $2l + 1 < r$) equals 0. For by (12)

$$t_1 + 2t_2 + 3t_3 + 4t_4 + \ldots + (2l + 1)\, t_{2l+1} = 2l + 1 .$$

Hence at least one of the integers $t_1, t_3, t_5, \ldots t_{2l+1}$ must be an *odd* number > 0. Hence at least one factor under the summation sign in (11) will be zero. To obtain k_r, r being odd, we note that every term of the summation (11) in which any one of the integers $t_1, t_3, t_5, \ldots, t_{r-2}$ is different from zero will vanish. Thus for the remaining terms $2t_2 + 4t_4 + 6t_6 + \ldots + rt_r = r$; whence t_r is not 0, and therefore

$$t_2 = t_4 = t_6 = \ldots = t_{r-1} = 0 , \; t_r = 1 .$$

$$\therefore\ k_r = -\frac{rA_{rs}}{r} = -A_{rs} .$$

Similarly, for k_{2l} we may write $t_1 = t_3 = t_5 = \ldots = t_{2l-1} = 0$. Whence we obtain by an easy reduction

$$k_{2l} = \Sigma \frac{(-r)^{t_2 + t_4 + t_6 + \ldots + t_{2l}}\, 3^{t_4} . 10^{t_6} . 35^{t_8} \ldots a_{2l-1}^{t_{2l}}}{\pi(t_2 + 1)\, \pi(t_4 + 1) \ldots \pi(t_{2l} + 1)\, 2^{t_4} . 3^{t_6} . 4^{t_8} \ldots l^{t_{2l}}} , \tag{18}$$

where
$$t_2 + 2t_4 + 3t_6 + \ldots + lt_{2l} = l .$$

Thus
$$k_2 = -r ;$$
$$k_4 = \frac{(-r)^2}{2} + \frac{3(-r)}{2} = \frac{r(r-3)}{1 . 2} ;$$
$$k_6 = \frac{(-r)^3}{2 . 3} + \frac{3(-r)^2}{2} + \frac{10(-r)}{3} = \frac{-r(r-4)(r-5)}{1 . 2 . 3} .$$

Substituting the values given by (16) and (17) in (13), we obtain

$$(2l)!\,k_{2l} = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 2r & 0 & 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2r & 0 & 3 & 0 & 0 & \dots & 0 \\ 6r & 0 & 2r & 0 & 4 & 0 & \dots & 0 \\ 0 & 6r & 0 & 2r & 0 & 5 & \dots & 0 \\ 20r & 0 & 6r & 0 & 2r & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot_{2l-1} \\ 2a_{2l-1}r & 0 & 2a_{2l-3}r & 0 & 2a_{2l-5}r & 0 & \dots & 0 \end{vmatrix}$$

Striking out the 1st row and 2d column; then the (present) 3d row and 4th column; then the 5th row and 6th column, etc., we have

$$= (-1)^l\,.\,1\,.\,3\,.\,5\,.\,7\dots(2l-1) \begin{vmatrix} 2r & 2 & 0 & 0 & \dots & 0 \\ 6r & 2r & 4 & 0 & \dots & 0 \\ 20r & 6r & 2r & 6 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2a_{2l-1}r & 2a_{2l-3}r & \dots & \dots & & 2r \end{vmatrix};$$

$$\therefore\ (-1)^l\,l!\,k_{2l} = \begin{vmatrix} r & 1 & 0 & 0 & 0 & \dots & 0 \\ 3r & r & 2 & 0 & 0 & \dots & 0 \\ 10r & 3r & r & 3 & 0 & \dots & 0 \\ 35r & 10r & 3r & r & 4 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot_{l-1} \\ a_{2l-1}r & a_{2l-3}r & a_{2l-5}r & \dots & \dots & \dots & r \end{vmatrix}. \qquad 19)$$

By either method we obtain the equation whose roots are the chords A_s, $-A_{m-s}$, $-A_{m+s}$, A_{2m-s}, A_{2m+s}, $\dots$, $(-1)^{\frac{1}{2}(r-1)}A_{\frac{1}{2}(r-1)m\pm s}$; viz.,

$$x^r - rx^{r-2} + \frac{r(r-3)}{1\,.\,2}x^{r-4} - \frac{r(r-4)(r-5)}{1\,.\,2\,.\,3}x^{r-6} + \frac{r(r-5)(r-6)(r-7)}{1\,.\,2\,.\,3\,.\,4}x^{r-8}$$

$$-\frac{r(r-6)(r-7)(r-8)(r-9)}{1\,.\,2\,.\,3\,.\,4\,.\,5}x^{r-10} + \dots + (-1)^{\frac{1}{2}(r-1)}\,.\,rx - A_{rs} = 0. \qquad (20)$$

If r be prime to m, the regular polygon of mr sides depends for inscription upon the same equations as does the regular r-gon, together with equations whose degrees are given by the prime factors of $m - 1$. If, however, r contains the factor m, the regular polygon of mr sides depends for inscription upon the same equations as does the regular r-gon, together with an equation of the mth degree of the form (20).

Compare Art. 109 of Serret's *Cours d'Algèbre Supérieure*.

University of Chicago, *December*, 1894.

DETERMINATION OF A CONIC FROM GIVEN CONDITIONS.

By Dr. J. Harrington Boyd, Chicago, Ill.

In this paper is given a method for the determination of a conic when the coordinates of one of its foci, the abscissa of the other, and the condition that the conic touches two given parallel lines are given.*

Take the given focus as the origin, then, if the center of the conic lie on the x-axis the equation to the conic in rectangular coordinates will be

$$x^2 + y^2 = e^2(\sigma + x)^2 \tag{1}$$

where e is the eccentricity and σ the distance from the origin to the directrix.

The equation of the conic, when its axes are revolved through an arbitrary angle λ, may be found by putting for x and y, respectively,

$$x \cos \lambda - y \sin \lambda \text{ and } x \sin \lambda + y \cos \lambda ; \tag{2}$$

whence

$$x^2 + y^2 = e^2(\sigma + x \cos \lambda - y \sin \lambda)^2 . \tag{3}$$

The major axis of the conic will now pass through the origin making an angle λ with the x-axis.

Represent the two given lines which are parallel to the y-axis by the equations

$$x = -l \text{ and } x = +l' . \tag{4}$$

If the conic represented by (3) touches the line $x = -l$, then the following quadratic equation must have equal roots :—

$$(1 - e^2 \sin^2\lambda)\, y^2 + 2e^2(\sigma - l \cos \lambda) \sin \lambda \,.\, y + l^2 - e^2\sigma^2 - e^2l^2 \cos^2\lambda$$
$$+ 2e^2 l\sigma \cos \lambda = 0 . \tag{5}$$

The condition that (5) have equal roots is

$$e^4(\sigma - l \cos \lambda)^2 \sin^2\lambda = (1 - e^2 \sin^2\lambda)(l^2 - e^2\sigma^2 - e^2l^2 \cos^2\lambda + 2e^2 l\sigma \cos \lambda)$$

or

$$l^2 = e^2(\sigma^2 + l^2 - 2\sigma l \cos \lambda) . \tag{6}$$

Similarly the condition that the conic in (3) touches the line $x = +l'$ will be

$$l'^2 = e^2(\sigma^2 + l'^2 + 2\sigma l' \cos \lambda) . \tag{7}$$

* This problem is interesting on account of its applicability in the determination of orbits of double stars, and was suggested, through Dr. See, by Prof. Burnham of the Yerkes Observatory.

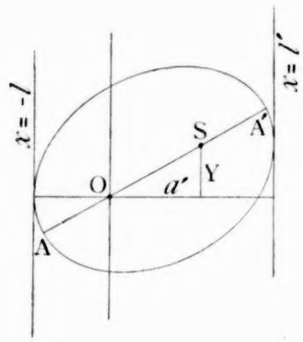
The third equation needed to determine the unknown quantities e, l, and λ is found in the following manner :—

The equation to the major axis of the conic may be written in the form

$$\frac{x}{\cos\lambda} = \frac{y}{\sin\lambda} = \rho\,, \tag{8}$$

where ρ is the distance of the point (x, y) from the origin O.

Substituting the values of x and y from (8) in (3) we get the following equation in ρ, whose roots are the distances AO and $A'O$ (see figure).



Here

$$\pm\rho = e\sigma + e\rho\cos 2\lambda\,, \tag{9}$$

whence

$$OA = \frac{e\sigma}{1 - e\cos 2\lambda} \quad \text{and} \quad -OA' = \frac{e\sigma}{1 + e\cos 2\lambda}\,,$$

$$AA' = \frac{2e\sigma}{1 - e^2\cos^2 2\lambda}\,, \quad OS = \frac{2e^2\sigma}{1 - e^2\cos^2 2\lambda}\,. \tag{10}$$

Consider now the coordinates of the focus S to be (a', Y), then

$$Y = OS\sin\lambda = OM\tan\lambda\,,$$

or

$$\frac{2e^2\sigma}{1 - e^2\cos^2 2\lambda} = \frac{a'}{\cos\lambda}\,. \tag{11}$$

We have now equations (6), (7), and (11) to solve for e, σ, and λ. Eliminating e^2 first from (6) and (7), then from (6) and (11) we get the following equations :

$$(l' - l)\,\sigma = 2ll'\cos\lambda\,, \quad \cos\lambda = a\sigma\,, \quad a = \frac{l' - l}{2ll'}\,,$$

$$2l\,(a' + l)\cos\lambda\sigma + l^2a'\cos^2 2\lambda - a'\sigma^2 - a'l^2 = 0\,. \tag{12}$$

Eliminating $\cos\lambda$ from the equations in (12),

$$\sigma = \pm \frac{1}{2la^2}\sqrt{1 - 2la\left[1 + \frac{l}{a'}\right] + 4a^2l^2}\,; \tag{13}$$

and, therefore,

$$\cos\lambda = \pm \frac{1}{2la}\sqrt{1 - 2la\left[1 + \frac{l}{a'}\right] + 4a^2l^2}\,.$$

The last equation in (13) enables one to draw the major axis of the conic. It follows from equation (11) that the expression for the eccentricity is

$$e^2 = \frac{a'}{2\sigma\cos\lambda + a'\cos^2 2\lambda}\,. \tag{14}$$

So, when $\cos\lambda$ and σ have been calculated by means of (13), then e can be found from (14) and the major axis from

$$AA' = \frac{2e\sigma}{1 - e^2\cos^2 2\lambda}\,.$$

The two signs in (13) correspond to the two positions which the conic may occupy with respect to the x-axis, namely, above and below it.

University of Chicago, *February* 20, 1895.

SOLUTIONS OF EXERCISES.

348

PROVE that, if $0 < \alpha < \beta$,

$$\int_\alpha^\beta \log \frac{\beta - x}{x - \alpha} \frac{dx}{x} = \frac{1}{2}\left[\log \frac{\beta}{\alpha}\right]^{2*}.$$

[*Frank Morley.*]

SOLUTION.

The integration is best effected by use of the following theorem, or lemma ·

$$\int_\alpha^\beta \varphi\left[\frac{x}{\beta}\right]\frac{dx}{x} = \int_\alpha^\beta \varphi\left[\frac{\alpha}{x}\right]\frac{dx}{x},$$

where $\varphi(z)$ is any finite and continuous function of z; and β and α have any fixed values, provided when one is zero the other must be infinite.

To prove this, let $\frac{x}{\beta} = \frac{\alpha}{y}$; whence

$$\frac{dx}{x} = -\frac{dy}{y}.$$

$$\therefore \int_\alpha^\beta \varphi\left[\frac{x}{\beta}\right]\frac{dx}{x} = -\int_\beta^\alpha \varphi\left[\frac{\alpha}{y}\right]\frac{dy}{y} = \int_\alpha^\beta \varphi\left[\frac{\alpha}{x}\right]\frac{dx}{x}.$$

Now

$$I \quad \int_\alpha^\beta \log\frac{\beta - x}{x - \alpha}\frac{dx}{x} = \int_\alpha^\beta \log\left\{\frac{\beta\left[1 - \frac{x}{\beta}\right]}{x\left[1 - \frac{\alpha}{x}\right]}\right\}\frac{dx}{x};$$

whence

$$I = \int_\alpha^\beta \log\frac{\beta}{x}\frac{dx}{x} + \int_\alpha^\beta \log\left[1 - \frac{x}{\beta}\right]\frac{dx}{x} - \int_\alpha^\beta \log\left[1 - \frac{\alpha}{x}\right]\frac{dx}{x}.$$

By the above lemma, the last two integrals are equal. We have, also,

$$\frac{dx}{x} = -d\log\frac{\beta}{x};$$

whence

$$I = -\int_\alpha^\beta \log\frac{\beta}{x}\, d\left[\log\frac{\beta}{x}\right] = \frac{1}{2}\left[\log^2\frac{\beta}{x}\right]_\beta^\alpha,$$

* The index, 2, is omitted in the text; it is undoubtedly an error on the part of the printer.

and

$$\int_{\alpha}^{\beta} \log \frac{\beta - x}{x - \alpha} \frac{dx}{x} = \tfrac{1}{2} \left[\log \frac{\beta}{\alpha} \right]^2 .$$

[*H. L. Rice.*]

380

THE sides of a variable rectangle pass through four fixed points. Find the position of the rectangle and its dimensions when its area is a maximum.

[*Geo. R. Dean.*]

SOLUTION.

Let A, B, C, D be the four points in order; a, b the diagonals AC, BD; and α the angle between them. Then if φ be inclination of a side p to AC, the two sides are

$$p = a \cos \varphi , \qquad q = b \sin (\varphi + \alpha) ;$$

and the area

$$pq = ab \sin (\varphi + \alpha) \cos \varphi = \tfrac{1}{2}\, ab \left[\sin (2\varphi + \alpha) + \sin \alpha \right]$$

is a maximum when $\varphi = 45^\circ - \tfrac{1}{2}\alpha$. The sides are

$$\tfrac{1}{2}\, a\sqrt{2}\, (\sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha) , \quad \tfrac{1}{2}\, b\sqrt{2}\, (\sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha) ,$$

and are equally inclined to the bisectrices of the angles between AC and BD.

[*Wm. M. Thornton.*]

Solved also by H. L. Rice and G. B. M. Zerr.

381

FROM a point in the circumference of a circle of radius R as centre is described the external arc of a circle of radius r. Determine r so that the area of the lune shall equal that of the original circle. [*W. M. Thornton.*]

SOLUTION.

Let φ be the angle at the center of the R-circle which subtends the chord

$$r = 2R \sin \tfrac{1}{2} \varphi .$$

Equating the area of the lune to that of the R-circle, we have

$$\tfrac{1}{2}\, r^2 (\pi + \varphi) - R^2\varphi + rR \cos \tfrac{1}{2} \varphi = \pi R^2 ,$$

whence

$$\tan \varphi = \pi + \varphi .$$

From which the value of φ, determined by trial, is

$$\varphi = 77^\circ\, 27'\, 12''.08$$

Whence

$$r = 1.25122\, R .$$

[*W. H. Echols.*]

Solved also by G. B. M. Zerr and H. L. Rice.

382

Four equal circles tangent to each other cut off equal areas from a given circle. Required the radii of the cutting circles when the aggregate area cut off from the given circle is the greatest possible. [*Artemas Martin.*]

SOLUTION.

The distance between the centres of the fixed circle and cutting circle $= r\sqrt{2}$, where r is the radius of a cutting circle. Let $R =$ radius of given circle. Then

$$4r^2 \cos^{-1}\left\{\frac{3r^2 - R^2}{2\sqrt{2}\,r^2}\right\} + 4R^2 \cos^{-1}\left\{\frac{R^2 + r^2}{2\sqrt{2}\,Rr}\right\} - 2\sqrt{6R^2r^2 - R^4 - r^4}$$

is the area cut off by the four circles. Differentiating and reducing, we get

$$r^2 \cos^{-1}\left\{\frac{3r^2 - R^2}{2\sqrt{2}r^2}\right\} = \tfrac{1}{2}\sqrt{6R^2r^2 - R^4 - r^4}\,. \tag{1}$$

Let

$$\cos\theta = \frac{3r^2 - R^2}{2\sqrt{2}r^2}\,. \tag{2}$$

Whence (1) becomes $\theta = \sqrt{2}\sin\theta$, the solution of which gives $\theta = 79° \, 43' \, 46''$. From (2),

$$r = .633\,R\,.$$

(1) is also satisfied by $r = (\sqrt{2} + 1)\,R$, and $r = (\sqrt{2} - 1)\,R$. The first of these values gives the four tangent circles circumscribed to the given circle, the second, the four tangent circles inscribed in the given circle.

[*G. B. M. Zerr.*]

383

If c', c'', c''' be the sides of any triangle inscribed in an ellipse, and b', b'', b''' the semi-diameters parallel to the sides, show that the area is

$$A = abc'c''c'''/(4b'b''b''')\,. \quad \text{[\textit{W. O. Whitescarver.}]}$$

SOLUTION I.

It is shown by Salmon (Conic Sections, p. 220) that if the triangle be given by the eccentric angles α, β, γ, its area is

$$= 2ab \sin\tfrac{1}{2}(\alpha - \beta)\sin\tfrac{1}{2}(\beta - \gamma)\sin\tfrac{1}{2}(\gamma - \alpha)\,.$$

He also shows on page 219 that

$$c' = 2b'\sin\tfrac{1}{2}(\alpha - \beta)\,, \quad c'' = 2b''\sin\tfrac{1}{2}(\beta - \gamma)\,, \quad c''' = 2b'''\sin\tfrac{1}{2}(\gamma - \alpha)\,;$$

whence by simple substitution we have the result above.

SOLUTION II.

The radius of the circumscribing circle is $\frac{b'b''b'''}{ab}$ (Salmon, p. 220); whence, at once,

$$A = \frac{abc'c''c'''}{4b'b''b'''}. \qquad \text{[\textit{W. O. Whitescarver.}]}$$

Solved also by G. B. M. Zerr, H. L. Rice, and F. G. Radelfinger.

384

If c be a chord of an ellipse through the points whose eccentric angles are α and β, and b' the semi-diameter parallel to the chord, show that the area of the triangle formed by the chord and the tangents at its extremities is

$$S = \frac{abc^2}{4b'^2} \tan \tfrac{1}{2} (\alpha - \beta). \qquad \text{[\textit{W. O. Whitescarver.}]}$$

SOLUTION.

The equations of the two tangents are

$$\frac{x \cos \alpha}{a} + \frac{y \sin \alpha}{b} = 1,$$

$$\frac{x \cos \beta}{a} + \frac{y \sin \beta}{b} = 1.$$

Their intersection has for its coordinates

$$x = \frac{a \cos \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)}, \quad y = \frac{b \sin \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)}.$$

$$\therefore \frac{2S}{ab} = \begin{vmatrix} \cos \alpha & \sin \alpha & 1 \\ \cos \beta & \sin \beta & 1 \\ \dfrac{\cos \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)} & \dfrac{\sin \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)} & 1 \end{vmatrix},$$

or

$$\frac{S}{ab} = -\sin \tfrac{1}{2} (\alpha - \beta) \cos \tfrac{1}{2} (\alpha - \beta) + \cos^2 \tfrac{1}{2} (\alpha + \beta) \tan \tfrac{1}{2} (\alpha - \beta)$$
$$+ \sin^2 \tfrac{1}{2} (\alpha + \beta) \tan \tfrac{1}{2} (\alpha - \beta).$$

$$\therefore \quad S = ab \tan \tfrac{1}{2} (\alpha - \beta) \sin^2 \tfrac{1}{2} (\alpha - \beta),$$

or, by Exercise 383,

$$S = \frac{abc^2}{4b'^2} \tan \tfrac{1}{2} (\alpha - \beta). \qquad \text{[\textit{H. L. Rice.}]}$$

Solved also by W. O. Whitescarver, F. G. Radelfinger, and G. B. M. Zerr.

385

If in exercise 384 a tangent be drawn parallel to the chord, show that the base of the triangle formed will be $c \sec \frac{1}{2} (\alpha - \beta)$, and its area $ab \tan^3 \frac{1}{2} (\alpha - \beta)$. [*W. O. Whitescarver.*]

SOLUTION.

If the base be t, we see that it touches the ellipse at the point $\gamma = \frac{1}{2} (\alpha + \beta)$ (see Salmon, Conic Sections, p. 219). Therefore Δ the area of triangle is, from Exercise 373, by substituting for γ,

$$\Delta = ab \tan \tfrac{1}{2} (\alpha - \beta) \tan^2 \tfrac{1}{4} (\alpha - \beta) \qquad (1)^*$$

or

$$= 2ab \frac{\tan^3 \frac{1}{4} (\alpha - \beta)}{1 - \tan^2 \frac{1}{4} (\alpha - \beta)}.$$

Since this triangle and that of Ex. 384 are similar we have $\frac{\Delta}{S} = \frac{t^2}{c^2}$, which gives

$$t = 2b' \tan \tfrac{1}{4} (\alpha - \beta). \qquad (2)^*$$

[*W. O. Whitescarver.*]

Solved also by H. L. Rice and G. B. M. Zerr.

392

A system of great circles intersect upon the equator of a sphere; a curve is drawn connecting points on the spherical surface where the circles of this system make a constant angle α with the meridians. Show that the stereographic projection of any such curve is a circular cubic whose equation may be written

$$\tan \alpha \tan \theta = \frac{c^2 + r^2}{c^2 - r^2},$$

c being the radius of the sphere, and one of the poles being the centre of the projection. [*R. A. Harris.*]

SOLUTION.

In stereographic projection the distance of any point from the origin is equal to the radius of the sphere multiplied by the tangent of one-half the polar distance, or $r = c \tan \frac{1}{2} \varphi$; whence

$$\sec \varphi = \frac{r^2 + c^2}{r^2 - c^2}.$$

If α is the constant angle, and θ the complement of the angle between the given meridian and the meridian of the moving point, we have

$$\sec \varphi = \tan \alpha \tan \theta.$$

[*Geo. R. Dean.*]

* The values given in the problem are wrong. The error came by mistaking $\frac{1}{4}$ for $\frac{1}{2}$ in equation (1). This of course changed (2).

EXERCISES.

396

SHOW how to solve the simultaneous equations:

1) $$x = y\sin(z + a) = y\sin z - a = y\sin(z + \beta) - b\,;$$

2) $$x(1 - \sin y) = a\,, \quad x[1 - \sin(y + \beta)] = b\,.$$

[*W. M. Thornton.*]

397

GIVEN two straight lines referred to rectangular coordinates; find geometrically an abscissa such that the sum of the squares of the two corresponding ordinates shall be a minimum. [*R. A. Harris.*]

398

IF any curve, plane or gauche, be represented by means of the coordinates P, Q, which are so connected with a set of orthogonal and isothermal coordinates p, q that $P =$ function p, $Q =$ function q, then the angles made by this curve and the curves, $Q =$ constant, $P =$ constant, are

$$\tan^{-1}\frac{dq}{dp}\,, \quad \tan^{-1}\frac{dp}{dq}\,,$$

respectively. [*R. A. Harris.*]

399

IF y be any cubic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula

$$\tfrac{1}{6}(y_1 - 2y_2 + y_3)(1 + 2\sin^2\varphi) + \tfrac{1}{2}(y_3 - y_1)\sin\varphi + y_2\,,$$

where

$$x_2 = \frac{h}{\sqrt{2}}\sec\varphi\sin(45^\circ - \varphi)\,, \quad x_3 = x_2 + \tfrac{1}{2}h\sec\varphi\,, \quad x_1 = x_2 - \tfrac{1}{2}h\sec\varphi\,.$$

[*W. H. Echols.*]

400

IF y be any cubic function of x, its mean value in any interval $X_2 - X_1 = L$ can be expressed in an infinite number of ways in terms of only two ordinates by the formula

$$\frac{3\sin^2\varphi}{1 + 2\sin^2\varphi}\,y_3 + \left[1 - \frac{3\sin^2\varphi}{1 + 2\sin^2\varphi}\right]y_2 + \tfrac{1}{18}L^3\,\frac{1 - 4\sin^2\varphi}{4\sin^2\varphi}\tan\varphi\,,$$

where

$$x_2 = X_1 + \frac{1}{\sqrt{2}}L\sec\varphi\sin(45^\circ - \varphi)\,, \quad x_3 = x_2 + \tfrac{1}{2}L\sec\varphi\,.$$

[*W. H. Echols.*]

401

DETERMINE the forms of surfaces which are such that the areas of parallel sections are cubic functions of their distances from a given section.

[*W. H. Echols.*]

402

Two circular arcs are tangent to each other and to the sides a and b of a triangle ABC at the points A and B. Show that the difference of curvature of the arcs is least when their common tangent makes with a and b the angles $\frac{1}{2}(3B - A)$ and $\frac{1}{2}(3A - B)$, respectively. [*W. H. Echols.*]

403

IF a triangle be inscribed in an ellipse, by drawing chords parallel to the tangents at the points α, β, γ; prove that the sides are

$$c' = 2b' \sin(\gamma - \beta), \quad c'' = 2b'' \sin(\alpha - \gamma), \quad c''' = 2b''' \sin(\beta - \alpha),$$

where b', b'', b''' are the semi-diameters conjugate to those at the points α, β, γ. Also the area of the triangle $= 2ab \sin(\alpha - \gamma) \sin(\beta - \alpha) \sin(\gamma - \beta)$.

[*G. B. M. Zerr.*]

404

THE area of a rectangle circumscribing an ellipse is

$$\frac{4ab}{b'^2}\left[(a^2 + b^2)\, b'^2 - a^2b^2\right]^{\frac{1}{2}},$$

where b' is the semi-diameter conjugate to that at the point of tangency of one of the sides. [*G. B. M. Zerr.*]

405

A TRAPEZOID circumscribing an ellipse is formed by the tangents at the extremities, and the tangents parallel to a chord passing through the points whose eccentric angles are α and β. Prove that its area $= 2ab \tan\theta \operatorname{cosec}^2\theta$, where $\theta = \frac{1}{4}(\alpha - \beta)$. [*G. B. M. Zerr.*]

406

IF y be any cubic function of x the mean value of y in any interval L is*

$$y_m + \frac{L^2}{24d^2}(y_1 + y_2 - 2y_m),$$

wherein y_m is the ordinate at the middle of the interval and the ordinates y_1 and y_2 are at a distance d on either side of y_m.

If x_m be the abscissa of the middle point of the interval, then in the limit when d vanishes we also have for the mean ordinate of the cubic $y = f(x)$,

$$y_m + \frac{L^2}{24} f''(x_m).$$

[*W. H. Echols.*]

* A generalization of Newton's rule.

CONTENTS.

Page.

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